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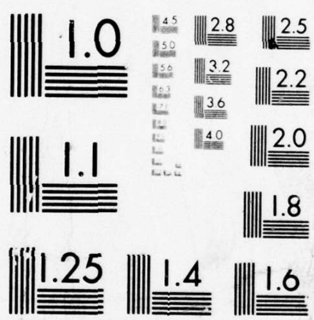

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Industrial & Systems
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Gainesville, FL. 32611

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OBSERVATIONS ON THE MINIMUM SPHERE PROBLEM

Research Report No. 77-5

by

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May 1977

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For a subset A of Euclidean n-space, the location problem of finding the point $\underline{x}$ which minimizes the maximum distance $d(\underline{x}, \underline{a})$ for $\underline{a}$ in A may be interpreted as finding the smallest sphere which encloses A. Algorithms have previously been developed for the situations when A is finite or a polytope. Here we concentrate on interpretations and properties of the problem, particularly its relation to other problems: cases where the sphere problem is dual to that of finding a shortest vector in A, where there is a connection with a

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20. Abstract (cont)

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ABSTRACT

For a subset A of Euclidean n-space, the location problem of finding the point $\underline{x}$ which minimizes the maximum distance $d(\underline{x}, \underline{a})$ for $\underline{a}$ in A may be interpreted as finding the smallest sphere which encloses A. Algorithms have previously been developed for the situations when A is finite or a polytope. Here we concentrate on interpretations and properties of the problem, particularly its relation to other problems: cases where the sphere problem is dual to that of finding a shortest vector in A, where there is a connection with a maximum moment-of-inertia problem, etc. Conjectures, relating the minimum sphere to points in A which define the diameter of A, are also discussed.

1. Introduction

For a subset S of E^n, the location problem of finding the point x which minimizes the maximum of $||x - a||$ for $a \in S$, may be interpreted as finding the smallest sphere which encloses S. Previously, algorithms have been developed for solving this problem when S is finite or a polytope [4, 6]. In this note we concentrate on interpretations and properties of the problem; particularly its relation to other problems. For example, we show a case where the sphere problem is dual to the problem of finding the vector of minimum norm in S. As another example (in E^2) we show the relation to a maximum moment of inertia problem. Other aspects discussed concern conjectures relating the minimum sphere to points in S which define the diameter of S.

2. Primal and Dual Formulation

The minimum sphere for a given finite set of points $a_i \in E^n$, $i = 1, 2, \ldots, m$, may be determined by solution of the mathematical program

$$\text{(P1)} \qquad \min_{(r,c)} \quad r$$

$$\text{s.t.} \quad r \geqq ||c - a_i|| \qquad i = 1, 2, \ldots, m$$

with variables r and c. The solution of (P1), (r^*, c^*), gives the radius and center of the sphere (the existence and uniqueness of (r^*, c^*), for any bounded set is guaranteed [1]). Problem (P1) is equivalent to

$$\text{(P2)} \qquad \min \quad s$$

$$\text{s.t.} \quad s \geqq ||c - a_i||^2 \qquad i = 1, 2, \ldots, m$$

where $s = r^2$. By introducing the variable $z = s - c^T c$[9], (P2) becomes the convex quadratic program

$$\text{(P3)} \qquad \min_{(z,c)} \quad z + c^T c$$

$$\text{s.t.} \quad z + 2a_i^T c \geqq a_i^T a_i \qquad i = 1, 2, \ldots, m$$

The dual of (P3) is the concave quadratic program [4]:

$$\text{(D3)} \qquad \max_{v} \quad h(v) = \sum v_i (a_i^T a_i) - v^T A^T A v$$

$$\text{s.t.} \quad \sum v_i = 1 \quad v_i \geqq 0 \qquad i = 1, 2, \ldots, m$$

where A has columns a_i.

A short note by Kuhn [8] shows that these problems (P3) and (D3) are "hybrid" programs, i.e., they are linear programs with a sum of squares added to the objective function. (P3) has this form and so does (D3) if we define $w \equiv Av$ so the second term in the objective is $w^T w$.

If v^* solves (D3) and (s^*, c^*) solves (P2) then

$$s^* = h(v^*)$$

and

$$c^* = \sum v_i^* a_i .$$

Note that the latter equation and the conditions on v in (D3) express the intuitively obvious fact that c* is a convex combination of the a_i. The coefficients expressing the optimal combination are the Kuhn-Tucker multipliers which solve (D3). By a theorem of Caratheodory [10], the convex combination may always be expressed with at most $n + 1$ positive coefficients. These facts are easily visualized for $a_i \in E^2$. The minimum circle will either be defined by the two most distance points, or it will pass through the vertices of three points which define a non obtuse triangle. Of course, more points may lie on the circle, but two or three suffice to express the center as a convex combination.

3. Duality Interpretations

In this section we show that two problems, one in mechanical physics and the other a geometrical "least distance" problem, are dual to the minimum sphere problem.

The first of these might be called a maximum moment of inertia problem. (I am grateful to my colleague, E. J. Muth, for pointing out this problem and its relation to the sphere problem.)

It is a basic principle of physics that a body set into rotation will tend to rotate about its center of gravity. This happens, for example, when a frisbee is sent spinning through the air. The physical principle involved is that natural forces tend to minimize the rotational energy of the body by choosing the center of gravity as the axis of rotation.

Consider now the following problem: Suppose we wish to assign masses m_i to a finite set of points a_i on a body so as to maximize its moment of inertia. We assume that the a_i are connected by thin rods or in some other manner involving negligible mass. For example, the a_i may be points on a thin lamina. Without loss of generality, we may assume the total mass to be distributed among the points is equal to one. Then, since the formula for moment of inertia, I, is

$$I = \sum m_i r_i^2$$

where r_i is the distance from the point a_i to the axis of rotation, we may write the problem as

$$\text{(PI)} \qquad \max_{m_i} \; \sum m_i \, ||a_i - \bar{x}||^2$$

$$\text{s.t.} \; \sum m_i = 1 \qquad m_i \geq 0 \qquad i = 1, 2, \ldots, m$$

where $\bar{x}$ is the center of gravity. But owing to the minimizing feature of the center of gravity we may write

(PI1) $\max_{m_i} \quad \min_{x} \quad \sum m_i ||a_i - x||^2$

s.t. $\sum m_i = 1, \quad m_i \geqq 0 \quad i = 1, 2, \ldots, m$

Thus the moment of inertia problem, (PI), is a maximin problem, and by a well known theorem [10, Corollary 37.3.2], the max min in (PI1) may be interchanged to min max. This yields

(DI1) $\min_{x} \quad \max_{m_i} \quad \sum m_i ||a_i - x||^2$

s.t. $\sum m_i = 1 \quad m_i \geqq 0 \quad i - 1, 2, \ldots, m.$

The inner maximization is easily solved by setting $m_j = 1$, where $||a_j - x||^2 = \max_i ||a_i - x||^2$, thus we obtain

(DI2) $\min \quad \max ||a_i - x||^2$

Which is equivalent to (P2) where $c = x$ and $c^* = \bar{x}$. Thus we have proven that *the solution of the problem (PI) is obtained by constructing the minimum circle for the given* a_i *and assigning the weights* $m_i = v_i^*$, *where* v^* *is the vector of optimal Kuhn-Tucker multipliers.*

It is interesting that the theorem of Caratheodory quoted earlier states that, for this problem, the maximum number of points with positive weight need be no greater than three. Also, the angular kinetic energy and the radius of gyration of a rotating body are proportional to the moment of inertia, so maximizing these quantities yields the same result.

The interpretation given above does not extend directly for $a_i \in E^3$ because the axis of rotation is a line rather than a point.

Our second problem arises in the minimization of nondifferentiable convex functions [13]. An important subproblem is to determine the vector, d*, of minimum norm in a given compact convex set S, i.e., solve the problem:

(PM) $\min ||d||^2$

s.t. $d \in S$.

If we make the assumption that S is a polytope with extreme points a_i, $i = 1, 2, \ldots, m$, then we have

(PM1) $$||d^*||^2 = \min_v ||\sum v_i a_i||^2$$

$$\text{s.t.} \quad \sum v_i = 1 \quad v_i \geqq 0 \quad i = 1, 2, \ldots, m.$$

If we *further assume that the a_i have the same Euclidean norm,* say $||a_i|| = 1$, then from (PM1) we obtain

$$1 - ||d^*||^2 = 1 - \min ||\sum v_i a_i||^2$$

$$= 1 + \max - ||\sum v_i a_i||^2$$

$$= \max 1 - ||\sum v_i a_i||^2$$

$$= \max \sum v_i(a_i{}^T a_i) - ||\sum v_i a_i||^2$$

$$\text{s.t.} \quad \sum v_i = 1, \; v_i \geqq 0.$$

Comparison with (D3) gives the (Pythagorean) dual relation

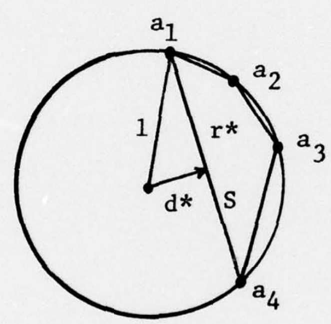
$$||d^*||^2 + s^* = 1$$

where s^* is the radius squared of the minimum sphere covering the a_i and $d^* = c^*$. See Figure 1 for a sketch. See [7] for computational experience where such normalizing of the a_i has proven helpful in nondifferentiable optimization. It would be interesting to know whether there is a geometrical interpretation for the case when not all $||a_i|| = 1$.

As an aside on problem (PM1), we note that it may be written as

$$\text{(PM2)} \qquad \max_{d} \, [\, - d^T d + 2 \min_{i} a_i^T d \,].$$

This formulation calls for the *unconstrained* maximization of a strictly concave function, but as far as we know, it has not been exploited in any solution procedure. To obtain (PM2) from (PM1), one has only to regard the latter as a quadratic programming problem and formulate Dorn's dual [3]. We omit the details of the derivation.



$||d^*||^2 + (r^*)^2 = 1$

Note: $(r^*)^2 = s^*$

FIGURE 1

4. Solution Methods

Since both (P3) and (D3) are quadratic programs, either could be solved by one of the well known algorithms designed for such problems. Care should be taken, however, because both objectives are *semidefinite*. Thus methods such as the Theil-Van de Panne procedure [12] which require a *definite* quadratic objective are not directly applicable, but others, such as Dantzig's procedure [2] are. If m is small, then (P3) or (D3) could both be solved with about the same effort. If, however, m is very large, (D3) is more attractive because there is only one constraint. In [4] Elzinga and Hearn give a decomposable solution procedure for (D3) which uses the Dantzig algorithm and for which the size of the tableau is independent of m. The basis for the decomposition is Caratheodory's theorem. A similar algorithm for the case when S is a polytope is described in [6].

For the problem in the plane one could also employ quadratic programming, but the following geometrical procedure quoted from [5] is more efficient:

"1. Pick any two given points and go to step 2.

2. Let the two points define the diameter of a circle. If this circle covers all points, stop. Otherwise choose some point outside the circle and the two defining points and go to step 3.

3. If the three points define a right triangle or an obtuse triangle, drop the point at the angle $\geq 90°$ and go to step 2 with the remaining two points. Otherwise if the triangle has strictly acute angles, go to step 4.

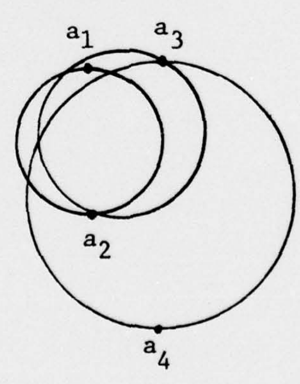
4. If the circle defined by the three points covers all points, stop. Otherwise choose some outside point, call it σ, and label as α a point from among the three defining points that is farthest from σ Extend the diameter of the current circle through point α to divide

the plane into two half-planes. Label as β the defining point in the same half-plane with σ and as γ the remaining point. With points α, γ, and σ, go to step 3.

In step 3 if the three points are on a straight line, we consider this a 'triangle' with a 180° angle at the intermediate point. Also note that by considering a right triangle as a two-point case we assume that in step 4 the point β is strictly contained in the half-plane with σ. In practice it seems reasonable to choose the farthest point outside the current circle rather than just some point as indicated in steps 2 and 3.

Convergence of this algorithm is based on showing that the circles created in each iteration are monotone increasing in radius. Then since there are only finitely many two-point and three-point circles, the process is finite."

We quote the algorithm here (but not the convergence proof) to emphasize (a) that it is initiated with an arbitrary pair of points, and (b) that once a point is covered, it need not remain covered (see Figure 2). All the procedure guarantees is that successive circles strictly increase in radius.



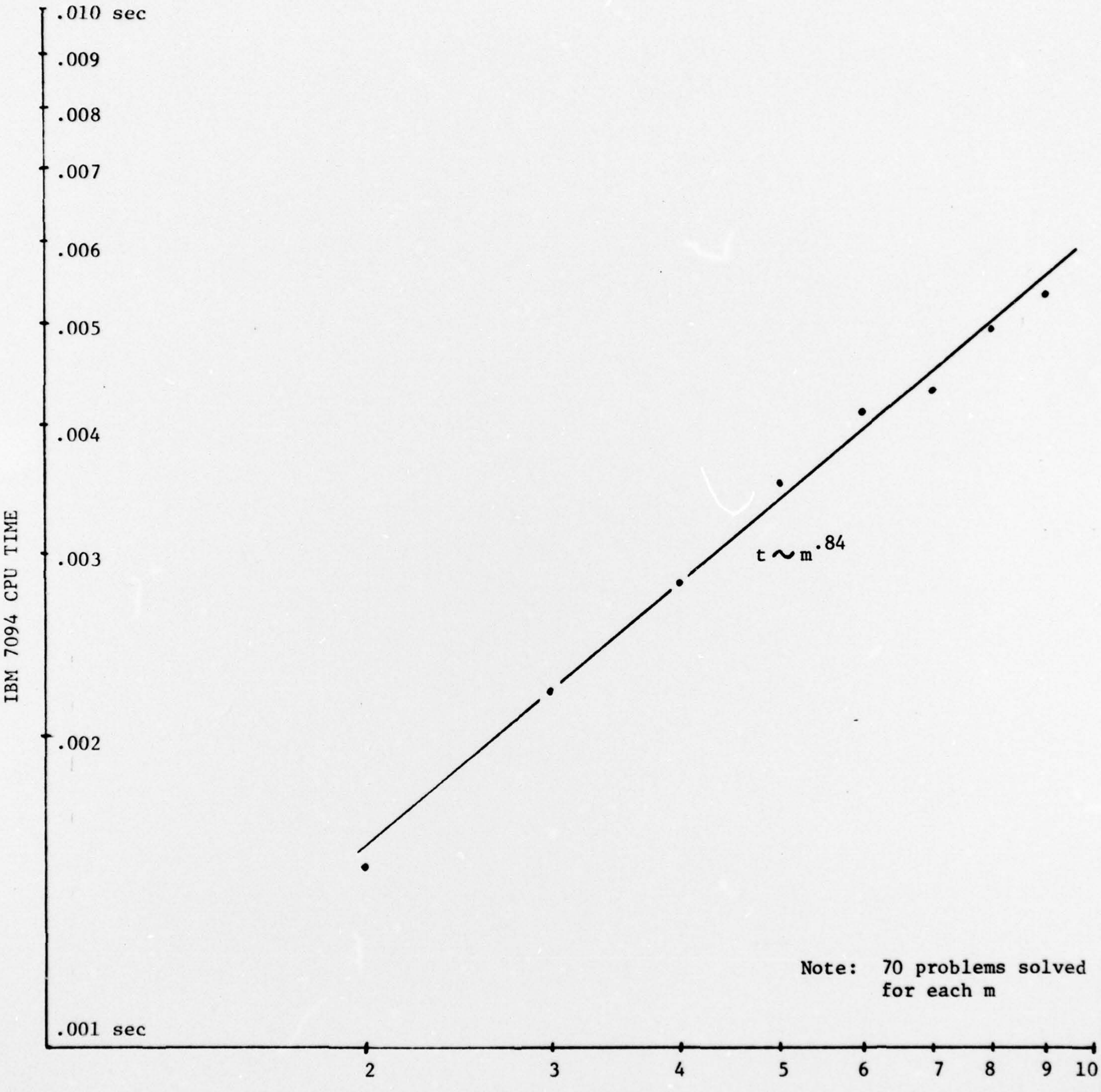
Point a_1 uncovered by third circle.

FIGURE 2

The efficiency of this geometrical algorithm, as measured by empirical testing, appears to be good for $m \leq 100$. (See Figures 3 and 4.) For large m, however, the time is probably of order m^2; in fact, this is suggested by Figure 4.

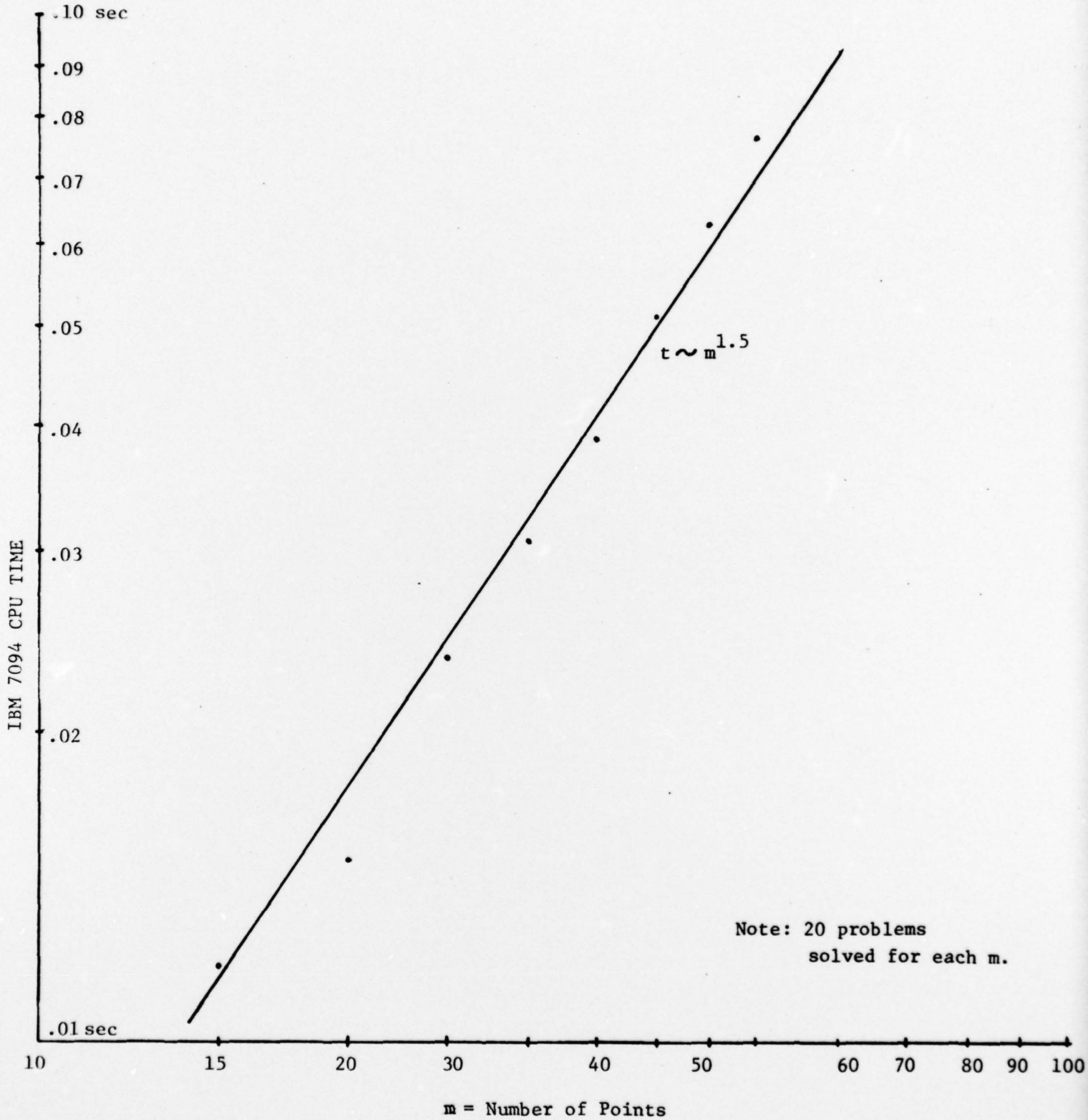
Whether a more efficient method can be devised is an open question. Certainly, an algorithm which kept all prior points covered would be of order m*, but would require that each circle be *efficiently* constructed from the previous one. In other words, the critical question would be how to construct the minimum circle for k + 1 points, knowing the minimum circle for k points. The answer is trivial for k + 1 = 2 or 3, but for k + 1 = 4, there are $\binom{4}{3} + \binom{4}{2} = 10$ possible circles to consider, and for larger k the subproblem seems no easier than the original.

* Shamos and Hoey [11] have conjectured that an algorithm of order $m \log_2 m$ is possible.



Time to Construct Minimum Circle

Figure 3



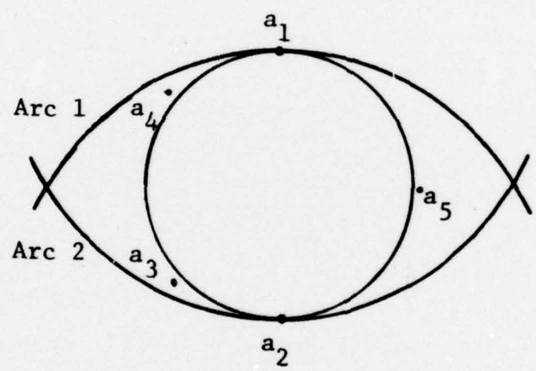
Time to Construct Minimum Circle

Figure 4

5. The Minimum Circle and Far Points

For a finite point set $S = \{a_1, a_2, \ldots, a_m\}$, $a_i \in E^2$, we define as *far points* any pair (a_k, a_ℓ) for which $||a_k - a_\ell|| = \max_{i,j} ||a_i - a_j||$ = diameter of S.

It is natural to conjecture that the minimum circle passes through at least one far point. Figure 5 represents a counterexample. The far points are a_1 and a_2 because the two arcs, Arc 1 and Arc 2 enclose all a_i. Clearly, however, the minimum circle is defined by a_3, a_4, and a_5. (The distances from a_5 to a_3 and a_4 are less than the diameter of the circle.)



Minimum Circle defined by
$\{a_3, a_4, a_5\}$.

FIGURE 5

The same figure enables us to determine the probability that a circle defined by two points will enclose a random third point.

Theorem Let a_1 and a_2 be fixed points and assume a random third point is no farther from a_1 and a_2 than $||a_1 - a_2||$. Then the probability P, that the circle defined by a_1 and a_2 covers the third point is

$$P = \frac{3\pi}{8 - 6\sqrt{3}} \doteq .64.$$

Proof: Consider Figure 6 where a_1 and a_2 (distance d apart) represent the two fixed points so that the third is enclosed in the area bordered by Arc 1 and Arc 2. The probability of it not being exterior to the circle defined by a_1 and a_2 is simply the ratio of the area of the circle to the area between the arcs.

Point a_2 is placed at the origin and the arcs each have a radius of d. We determine the area between the arcs by integration over one-quarter of the area (shaded portion):

$$\text{Area between arcs} = 4 \int_0^{\bar{x}} ((d^2 - x^2)^{1/2} - \frac{d}{2})\, dx,$$

where $\bar{x}$ is determined by

$$\frac{d^2}{4} + \bar{x}^2 = d^2 \Rightarrow \bar{x} = \pm \frac{d}{2} \sqrt{3}.$$

So,

$$\text{area between arcs} = 4 \int_0^{\frac{d}{2}\sqrt{3}} ((d^2 - x^2)^{1/2} - \frac{d}{2})\, dx$$

$$= 2 \left[x \sqrt{d^2 - x^2} + d^2 \sin^{-1} \frac{x}{d} \right]_0^{\frac{d}{2}\sqrt{3}}$$

$$-4d \left[\frac{x}{2} \right]_0^{\frac{d}{2}\sqrt{3}}$$

$$= 2((\frac{d}{2}\sqrt{3}) \sqrt{d^2 - \frac{3d^2}{4}} + d^2 \sin^{-1} \frac{\sqrt{3}}{2}) - d^2\sqrt{3}$$

$$= 2d^2 \left(\sqrt{\frac{3}{4}} + \sin^{-1} \sqrt{\frac{3}{2}} - \sqrt{\frac{3}{2}} \right)$$

$$= 2d^2 \left(\frac{\pi}{3} - \sqrt{\frac{3}{4}} \right) .$$

Since the area of the circle is $\pi \frac{d^2}{4}$, the ratio is

$$P = \frac{\pi}{8 \left(\frac{\pi}{3} - \sqrt{\frac{3}{4}} \right)} = \frac{3/2\ \pi}{4\pi - 3\sqrt{3}} = \frac{3\pi}{8\pi - 6\sqrt{3}} \doteq .64 .$$

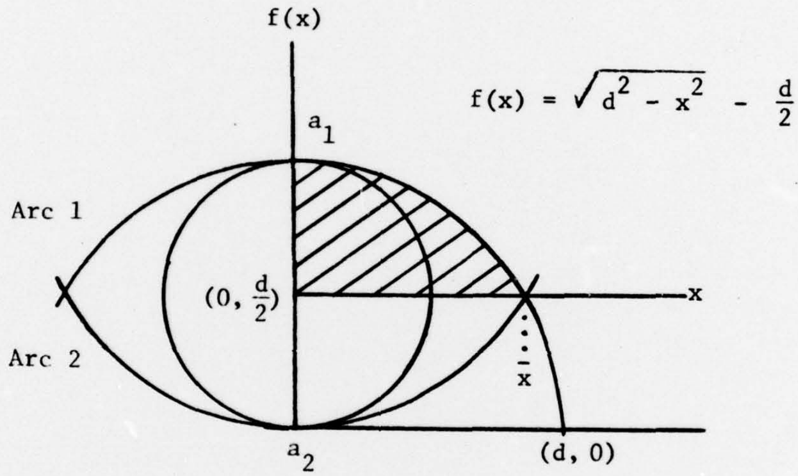


FIGURE 6

This result suggests further problems of the same sort. For example, what is the probability that three points uniformly distributed over some convex set in R^2 will form a triangle with an obtuse angle (so that the minimum circle is defined by the two far points)? When the set is the unit square, Monte Carlo simulation shows that the answer is approximately .70. To our knowledge, the question posed has not been answered analytically. For the interested researcher, the monograph by Kendall and Moran [14] is a starting point.

References

1. Blumenthal, L. M. and Wahlin, G. E., "On the Spherical Surface of Smallest Radius Enclosing a Bounded Subset of n - Dimensional Euclidean Space," American Mathematical Society Bulletin, Vol. 47 (1941).

2. Dantzig, G. B., Linear Programming and Extensions, Princeton University Press (1963).

3. Dorn, W. S., "Duality in Quadratic Programming," Quarterly of Applied Mathematics, 18 (1960).

4. Elzinga, J. and Hearn, D. W., "The Minimum Covering Sphere Problem," Management Science 19, 96-104 (1972).

5. Elzinga, J. and Hearn, D. W., "Geometrical Solutions for Some Minimax Location Problems," Transportation Science, Vol. 6, No. 4 (1972).

6. Elzinga, J. and Hearn, D. W., "The Minimum Sphere Covering a Convex Polyhedron," NRLQ, Vol. 21, No. 4 (1974).

7. Hearn, D. W. and Lowe, T. J., "A Subgradient Procedure for the Solution of Minimax Location Problems," to appear in Computers and Industrial Engineering.

8. Kendall, M. G. and Moran, P. A. P., "Geometrical Probability," Griffin's Statistical Monograph No. 10, Hafner Publishing Co., New York, 1963.

9. Kuhn, H. W., "Sylvester's Problem is a Hybrid Program," unpublished note, Princeton University, March 1975.

10. Nair, K. P. K. and Chandresekaran, R., "Optimal Location of a Single Service Center of Certain Types," NRLQ, 18 (1971).

11. Rockafellar, R. T., Convex Analysis, Princeton University Press, 1970.

12. Shamos, M. I. and Hoez, D. "Closest Point Problems," Proceedings 16th Annual Symposium on Foundations of Computer Science. (IEEE), Berkeley, Calif. (1975).

13. Theil, H. and van de Panne, C., "Quadratic Programming as an Extension of Conventional Quadratic Maximization," Management Science, Vol. 7 (1960).

14. Wolfe, P., "Algorithm for a Least-Distance Programming Problem," Mathematical Programming Study 1 (1974).